Uniform

$$f(y) = \frac{1}{\theta_2 - \theta_1} \quad \text{if } \theta_1 < y < \theta_2$$

Proof: $E[Y] = \dfrac{\theta_1 + \theta_2}{12}$

$$E[Y] = \int_{\theta_1}^{\theta_2} \frac{y}{\theta_2 - \theta_1}\, dy$$

$$= \frac{\frac{1}{2} y^2}{\theta_2 - \theta_1} \Bigg|_{\theta_1}^{\theta_2}$$

$$= \frac{\frac{1}{2}\theta_2^2}{\theta_2 - \theta_1} - \frac{\frac{1}{2}\theta_1^2}{\theta_2 - \theta_1}$$

$$= \frac{\frac{1}{2}\cancel{(\theta_2 - \theta_1)}(\theta_2 + \theta_1)}{\cancel{\theta_2 - \theta_1}} = \frac{1}{2}(\theta_2 + \theta_1) \qquad \square$$

Proof $Var(Y) = E[Y^2] - E[Y]^2$

$$E[Y^2] = \int_{\theta_1}^{\theta_2} \frac{y^2}{\theta_2 - \theta_1}\, dy$$

$$= \frac{\frac{1}{3} y^3}{\theta_2 - \theta_1} \Bigg|_{\theta_1}^{\theta_2}$$

$$= \frac{\frac{1}{3}\theta_2^3 - \frac{1}{3}\theta_1^3}{\theta_2 - \theta_1}$$

$$Var(Y) = \frac{\frac{1}{3}(\theta_2^3 - \theta_1^3)}{\theta_2 - \theta_1} - \frac{1}{2} * (\theta_2 + \theta_1)$$

$$= \frac{\theta_2^3 - \theta_1^3}{3(\theta_2 - \theta_1)} - \left(\frac{\theta_2 + \theta_1}{2}\right)^2$$

$$= \frac{4(\theta_2^3 - \theta_1)^3 - 3(\theta_2^2 + 2\theta_2\theta_1 + \theta_1^2)(\theta_2 - \theta_1)}{12(\theta_2 - \theta_1)}$$

$$= \frac{4(\theta_2^2 + \theta_2\theta_1 + \theta_1^2) - 3(\theta_2^2 + 2\theta_2\theta_1 + \theta_1^2)}{12}$$

$$= \frac{\theta_2^2 - 2\theta_2\theta_1 + \theta_1^2}{12} = \frac{(\theta_2^2 - \theta_1^2)^2}{12} \qquad \square$$

Gamma

$Y \sim \text{gamma}(\alpha, \beta)$

Proof $E[Y] = \alpha\beta$

$$LHS = \int_0^\infty y \cdot \frac{y^{\alpha-1} e^{-\frac{y}{\beta}}}{\Gamma(\alpha)\beta^\alpha}\, dy$$

$$= \int_0^\infty \frac{y^{(\alpha+1)-1} e^{-\frac{y}{\beta}}}{\Gamma(\alpha)\,\beta^\alpha}\, dy$$

let $k = \alpha + 1$

$$= \frac{\Gamma(k)\,\beta^k}{\Gamma(\alpha)\beta^\alpha} \underbrace{\int_0^\infty \frac{y^{k-1} e^{-\frac{y}{\beta}}}{\Gamma(k)\,\beta^k}\, dy}_{F \text{ of a gamma } (k,\beta)} = 1$$

$$= \frac{\Gamma(\alpha+1)\,\beta^{\alpha+1}}{\Gamma(\alpha)\,\beta^\alpha} = \frac{\alpha\Gamma(\alpha)\beta}{\Gamma(\alpha)} = \alpha\beta \quad \square$$

Proof $Var(Y) = \alpha\beta^2$

$LHS = E[Y^2] - E[Y]^2$

$$= \int_0^\infty y^2 \frac{y^{\alpha-1} e^{-\frac{y}{\beta}}}{\Gamma(\alpha)\,\beta^\alpha}\, dy - E[Y]^2$$

$$= \int_0^\infty \frac{y^{(\alpha+2)-1} e^{-\frac{y}{\beta}}}{\Gamma(\alpha)\,\beta^\alpha}\, dy - E[Y]^2$$

let $k = \alpha + 2$

$$= \frac{\Gamma(k)\,\beta^k}{\Gamma(\alpha)\beta^\alpha} \underbrace{\int_0^\infty \frac{y^{k-1} e^{-\frac{y}{\beta}}}{\Gamma(k)\,\beta^k}\, dy}_{F(\infty) \text{ of } \text{gamma}(k,\beta)} = 1 \qquad - E[Y]^2$$

$$= \frac{\Gamma(\alpha+2)\,\beta^{\alpha+2}}{\Gamma(\alpha)\,\beta^\alpha} - (\alpha\beta)^2 = \frac{(\alpha+1)\Gamma(\alpha+1)\,\beta^2}{\Gamma(\alpha)} - (\alpha\beta)^2 = \frac{(\alpha+1)(\alpha)\cancel{\Gamma(\alpha)}\,\beta^2}{\cancel{\Gamma(\alpha)}} - (\alpha\beta)^2$$

$$= \cancel{\alpha^2\beta^2} + \alpha\beta^2 - \cancel{\alpha^2\beta^2}$$

$$= \alpha\beta^2 \quad \square$$

Moments

Proof $Y \sim poisson(\lambda)$

$m_Y(t) = e^{\lambda(e^t-1)}$

LHS $= E[e^{tY}]$

$= \sum_{i=0}^{\infty} e^{ty} \times \frac{\lambda^y e^{-\lambda}}{y!}$

$= e^{-\lambda} \sum_{i=0}^{\infty} \frac{(e^t \lambda)^y}{y!}$

$= e^{-\lambda} \times e^{e^t \lambda}$

$= e^{\lambda(e^t-1)}$ $\square$

Proof $Y \sim gamma(\alpha, \beta)$

$m_Y(t) = (1-t\beta)^{-\alpha}$

LHS $= E[e^{tY}]$

$= \int_0^\infty e^{ty} \times \frac{y^{\alpha-1} e^{-y/\beta}}{\Gamma(\alpha)\beta^\alpha} dy$

$= \int_0^\infty \frac{y^{\alpha-1} e^{-y(\frac{1}{\beta}-t)}}{\Gamma(\alpha)\beta^\alpha} dy$

$= \frac{1}{\Gamma(\alpha)\beta^\alpha} \int_0^\infty y^{\alpha-1} e^{\frac{-y}{(\frac{1}{\beta}-t)^{-1}}} dy$

$\frac{1}{\beta} - t \neq 0$

$\therefore \frac{1}{\beta} \neq t$

let $k = (\frac{1}{\beta} - t)^{-1}$

$= \frac{\Gamma(\alpha) k^\alpha}{\Gamma(\alpha)\beta^\alpha} \underbrace{\int_0^1 \frac{y^{\alpha-1} e^{-\frac{y}{k}}}{\Gamma(\alpha) k^\alpha} dy}_{gamma(\alpha, k)}$

$= \frac{k^\alpha}{\beta^\alpha} = \frac{(\frac{1}{\beta}-t)^{-\alpha}}{\beta^\alpha} = \left(\frac{1}{\frac{\frac{1}{\beta}-t}{\beta}}\right)^\alpha$

$= \left(\frac{1}{1-\beta t}\right)^\alpha$

why?

$= (1-\beta t)^{-\alpha}$ $t < \frac{1}{\beta}$ $\square$

$Y \sim \text{normal}(\mu, \sigma^2)$

Proof $\quad m_Y(t) = e^{\mu t + \frac{1}{2}t^2\sigma^2}$

$LHS = E[e^{tY}]$

$$= \int_{-\infty}^{\infty} e^{ty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2} dy$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \times e^{ty - \frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2} dy$$

let $k = ty - \frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2$

$$= ty - \frac{y^2 - 2y\mu + \mu^2}{2\sigma^2}$$

$$= -\frac{y^2 - 2y\mu + \mu^2 - 2\sigma^2 ty}{2\sigma^2}$$

$$= -\frac{y^2 + \mu^2 - 2y(\mu + \sigma^2 t)}{2\sigma^2}$$

$$= \frac{-\left(y^2 - 2y(\mu+\sigma^2 t) + (\mu+\sigma^2 t)^2\right) - \mu^2 + (\mu+\sigma^2 t)^2}{2\sigma^2}$$

$$= \frac{-(y - \mu + \sigma^2 t)^2 - \mu^2 + (\mu + \sigma^2 t)^2}{2\sigma^2}$$

$$= \frac{-(y - \mu - \sigma^2 t)^2 + 2\mu\sigma^2 t + t^2\sigma^4}{2\sigma^2}$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\frac{(y-\mu-\sigma^2 t)^2}{\sigma^2}} e^{\mu t + \frac{1}{2}t^2\sigma^2} dy$$

$$= e^{\mu t + \frac{1}{2}t^2\sigma^2} \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{y-\mu-\sigma^2 t}{\sigma}\right)^2} dy$$

$$= e^{\mu t + \frac{1}{2}t^2\sigma^2} \quad \square$$

Tschebbyshetf's thm

Proof $P(|Y-\mu| \le k\sigma) \ge 1 - \frac{1}{k^2}$

$\sigma^2 = E[(Y-\mu)^2]$

$= \int_{-\infty}^{\infty} (Y-\mu)^2 f(y)\, dy$

$= \int_{-\infty}^{\mu-k\sigma} (Y-\mu)^2 f(y)\, dy + \underbrace{\int_{\mu-k\sigma}^{\mu+k\sigma} \overbrace{(Y-\mu)^2 f(y)}^{a}\, dy}_{A} + \int_{\mu+k\sigma}^{\infty} (Y-\mu)^2 f(y)\, dy$

$f(y) \ge 0, \ (Y-\mu)^2 \ge 0$

$\therefore a \ge 0 \quad \therefore A \ge 0$

$\ge \int_{-\infty}^{\mu-k\sigma} (Y-\mu)^2 f(y)\, dy + \int_{\mu+k\sigma}^{\infty} (Y-\mu)^2 f(y)\, dy$

$y < \mu - k\sigma$

$y - \mu < -k\sigma$

$\mu - y \ge k\sigma$

$(\mu-y)^2 f(y) \ge k^2\sigma^2 f(y)$

$\ge \int_{-\infty}^{\mu-k\sigma} k^2\sigma^2 f(y)\, dy + \int_{\mu+k\sigma} k^2\sigma^2 f(y)\, dy$

$= k^2\sigma^2 \left[\int_{-\infty}^{\mu-k\sigma} f(y)\, dy + \int_{\mu+k\sigma}^{\infty} f(y)\, dy\right]$

$= k^2\sigma^2 \left[P(y \le \mu - k\sigma) + P(Y \ge \mu + k\sigma)\right]$

$\frac{1}{k^2} \ge P(Y \le \mu - k\sigma) + P(Y \ge \mu + k\sigma)$

$\frac{1}{k^2} \ge P(Y - \mu \le -k\sigma) + P(Y - \mu \ge k\sigma)$

$\frac{1}{k^2} \ge P(|Y-\mu| \ge k\sigma)$

Covariance

Proof $cov(Y_1, Y_2) = E[Y_1 Y_2] - E[Y_1]E[Y_2]$

$$LHS = E[(Y_1 - \mu_1)(Y_2 - \mu_2)]$$
$$= E[Y_1 Y_2 - Y_2 \mu_1 - Y_1 \mu_2 + \mu_1 \mu_2]$$
$$= E[Y_1 Y_2] - \mu_1 E[Y_2] - \mu_2 E[Y_1] + \mu_1 \mu_2$$
$$= E[Y_1 Y_2] - E[Y_1] E[Y_2] \quad \square$$

Linear combinations

$$U = \sum_{i=1}^{n} a_i Y_i \qquad W = \sum_{j=1}^{m} b_j X_j$$

Proof $E[U] = \sum_{i=1}^{n} a_i \mu_i$

$$LHS = E\left[\sum_{i=1}^{n} a_i Y_i\right]$$
$$= \sum_{i=1}^{n} E[a_i Y_i]$$
$$= \sum_{i=1}^{n} a_i E[Y_i] = \sum_{i=1}^{n} a \mu_i \quad \square$$

Proof $var(U) = E[(U - E[U])^2]$

$$= E\left[\left(\sum_{i=1}^{n} a_i Y_i - \sum_{i=1}^{n} a_i \mu_i\right)^2\right]$$
$$= E\left[\left(\sum_{i=1}^{n} a_i (Y_i - \mu_i)\right)^2\right]$$
$$= E\left[\sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j cov(Y_i, Y_j)\right]$$
$$= E\left[\sum_{i=1}^{n} a_i a_i cov(Y_i, Y_i) + \sum_{i \neq j}^{n} \sum a_i a_j cov(Y_i, Y_j)\right]$$
$$= E\left[\sum_{i=1}^{n} a_i^2 var(Y_i) + 2\sum_{i=1}^{n} \sum_{j=1}^{i} a_i a_j cov(Y_i, Y_j)\right]$$
$$= \sum_{i=1}^{n} a_i^2 var(Y_i) + 2\sum_{i=1}^{n} \sum_{j=1}^{i} a_i a_j cov(Y_i, Y_j)$$

$$= \sum var(Y_i) + 2\sum_{1 < i < j} \sum a_i a_j Cov(Y_i, Y_j)$$

Proof $cov(U,V) = \sum_{i=1}^{n} \sum_{j=1}^{m} a_i b_j \, cov(Y_i, X_j)$

LHS: $E[(U - E(U))(V - E[V])]$

$$= E\left[\left(\sum_{i=1}^{n} a_i Y_i - \sum_{i=1}^{n} a_i E[Y_i]\right)\left(\sum_{j=1}^{m} b_j X_j - \sum_{j=1}^{m} b_j E[X_j]\right)\right]$$

$$= E\left[\left(\sum_{i=1}^{n} a_i (Y_i - E[Y_i])\right)\left(\sum_{j=1}^{m} b_j (X_j - E[X_j])\right)\right]$$

$$= E\left[\sum_{i=1}^{n} \sum_{j=1}^{m} a_i b_j (Y_i - E[Y_i])(X_j - E[X_j])\right]$$

$$= E\left[\sum_{i=1}^{n} \sum_{j=1}^{m} a_i b_j \, cov(Y_i, Y_j)\right]$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{m} a_i b_j \, cov(Y_i, Y_j)$$

Multinomial distribution

$(Y_1, Y_2, \dots, Y_k) \sim multinomial(n, p_1, p_2, \dots, p_k)$

$Y_i \sim binomial(n, p_i)$

$E[Y_i] = np_i$, $Var(Y_i) = np_i(1-p_i)$

Proof $cov(Y_i, Y_j) = -np_i p_j$

$$I_h = \begin{cases} 1 & \text{if trial } h \text{ in class } i \\ 0 & \text{else} \end{cases}$$

$E[I_h] = 1 \times p_i + 0 \times (1-p_i) = p_i$ $\qquad Var(I_h) = E[(I_h - E[I_h])^2]$
$= (1-p_i)^2 \times 1$

$$J_l = \begin{cases} 1 & \text{if trial } l \text{ in class } j \\ 0 & \text{else} \end{cases}$$

$E[J_l] = p_j$

$cov(I_h, J_l) = 0$ if $h \neq l$ $\qquad$ indep.

$cov(I_m, J_m) = E[I_m J_m] - E[I_m]E[J_m]$
$= 0 - p_i p_j = -p_i p_j$

$Y_i = \sum_{h=1}^{n} I_h \qquad Y_j = \sum_{l=1}^{n} J_l$

$$cov(Y_i, Y_j) = \sum_{h=1}^{n} \sum_{l=1}^{n} cov(I_h, J_l)$$

$$= \sum_{m=1}^{n} cov(I_m, J_m) + \sum_{h \neq l} \underbrace{cov(I_h, J_l)}_{0}$$

$$= n \times -p_i p_j + 0 \quad \square$$

$P(A|B) = \frac{P(AB)}{P(B)}$

Conditional distributions

Proof $E[Y_1] = E[E[Y_1|Y_2]]$

$$LHS = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y_1 f(y_1, y_2)\, dy_1\, dy_2$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y_1 \times f(y_1|y_2) \times f(y_2)\, dy_1\, dy_2$$

$$= \int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} y_1 f(y_1|y_2)\, dy_1\right\} f(y_2)\, dy_2$$

$$= \int_{-\infty}^{\infty} E[Y_1|Y_2]\, f(y_2)\, dy_2$$

$$= E[E[Y_1|Y_2]] \quad \square$$

Proof $Var(Y_1) = E[Var(Y_1|Y_2)] + Var[E[Y_1|Y_2]]$

$$LHS = E[Y_1^2] - E[Y_1]^2$$

$$= E[E[Y_1^2|Y_2]] - E[E[Y_1|Y_2]]^2 \underbrace{- E[E[Y_1|Y_2]^2] + E[E[Y_1|Y_2]^2]}_{\text{Remember!}}$$

$$= E[E[Y_1^2|Y_2] - E[Y_1|Y_2]^2] + E[E[Y_1|Y_2]^2] - E[E[Y_1|Y_2]]^2$$

$$= E[Var(Y_1|Y_2)] + Var(E[Y_1|Y_2])$$

Moment-gen-fnr

Proof $U = \sum_{i=1}^{n} Z_i^2 \sim \chi_n^2$ if $Z_i \underset{iid.}{\sim} n(0,1)$

$m_U(t) = E[e^{t\sum Z_i^2}]$

$= E[\prod e^{tZ_i^2}]$

$= \prod E[e^{tZ_i^2}]$ iid

$= \prod m_{\chi_1^2}(t)$

$= \prod (1-2t)^{-\frac{1}{2}}$

$= (1-2t)^{-\frac{n}{2}}$

$U \sim \chi_n^2$

Theorem 3.12

Proof $\left.\frac{d^k m(t)}{dt^k}\right|_{t=0} = \mu'_k$

$$m(t) = E[e^{tY}]$$
$$= E\left[\frac{(tY)^0}{0!} + \frac{(tY)^1}{1!} + \frac{(tY)^2}{2!} + \dots\right]$$
$$= 1 + E[tY] + \frac{E[t^2Y^2]}{2!} + \dots$$
$$= 1 + tE[Y] + \frac{t^2}{2!} \times E[Y^2] + \frac{t^3}{3!} E[Y^3] + \dots$$

$$\frac{d^1 m(t)}{dt^1} = E[Y] + tE[Y^2] + \frac{t^2}{2!} E[Y^3] + \dots$$

$$\left.\frac{d^1 m(t)}{dt^1}\right|_{t=0} = E[Y] = \mu'_1$$

$$\frac{d^2 m(t)}{dt^2} = E[Y^2] + tE[Y^3] + \dots$$

$$\frac{d^2 m(t)}{dt^2} = E[Y^2]$$

In general $\left.\frac{d^k m(t)}{dt^k}\right|_{t=0} = E[Y^k] = \mu'_k$ $\square$

Theorem 5.9.

Proof $E[g(Y_1)h(Y_2)] = E[g(Y_1)]\,E[h(Y_2)]$ if Y_1, Y_2 indep

$$LHS = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(y_1)h(y_2) \times f(y_1, y_2)\,dy_1\,dy_2$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(y_1)h(y_2) f(y_1) f(y_2)\,dy_1\,dy_2$$

$$= \int_{-\infty}^{\infty} h(y_2) f(y_2) \int_{-\infty}^{\infty} g(y_1) f(y_1)\,dy_1\,dy_2$$

$$= \int_{-\infty}^{\infty} h(y_2) f(y_2) E[g(Y_1)]\,dy_2$$

$$= E[h(Y_2)]\,E[g(Y_1)]$$ $\square$

Thm. 5.12.

Chi-squared

Proof: $\sum_{i=1}^{n} Z_i^2 \sim \chi_n^2$

let $U = \sum_{i=1}^{n} Z_i^2$

$$m_U(t) = E[e^{t\sum_{i=1}^{n} Z_i^2}]$$

$$= \prod_{i=1}^{n} E[e^{tZ_i^2}]$$

$$= \prod_{i=1}^{n} m_{Z_i^2}(t)$$

$$= \prod_{i=1}^{n} (1-2t)^{-\frac{1}{2}}$$

$$= (1-2t)^{-\frac{n}{2}}$$

$\therefore U \sim \chi_n^2$